Appendix

C

Cooperative Game Theory

IN THIS appendix we focus on a particular aspect of cooperative game theory associated with arbitrated solutions and issues of fairness and efficiency. Nevertheless, to position these issues in the game theory domain we start with an introduction of basic concepts, some of them applicable to all games. Then we discuss briefly the difference between the cooperative and non-cooperative games followed by a general description of different aspects of cooperative games. Finally we describe the arbitrated solutions in the context of fairness and efficiency.

C.1 Basic Concepts

C.1.1 Players

Every game has a set of players which can be considered rational decision makers. In general the set of players, $J = \{1, 2, ..., n\}$, is finite and each player knows how many players take part in the game. A random element of the game can be introduced by adding player 0, which is called nature of chance. Player 0 makes its decision based on a probability distribution which is known to all the other players. In the following three forms of the games are described: extensive, strategic, and coalitional.

C.1.2 Extensive form

The extensive form describes a full tree of possible decision sequences. Consider, for example, a game with two players, P1 and P2, where the first player starts the game by choosing one of the two options, L or R. Then the second player tries to guess which option was chosen by the opponent. If the choice is right, he wins one unit of money from the opponent; otherwise he pays one unit to the first player. The loosing player has right to start another matching or quit (Q). Assuming that the game is limited to two matchings, all possible game sequences and outcomes are illustrated in Figure C.1. The game moves from node to node along the tree. Although the tree form suggest that all actions are made in sequence, in fact the choice and matching are done in parallel. This is illustrated by the dashed boxes which indicate that the current information set is imperfect. In other words, the player who makes decision at this stage does not know in which particular node of the dashed box the game is located.

Utility functions

In Figure C.1, the outcomes of the game are indicated in monetary payoffs. Observe that the players' satisfaction from the game may be not proportional to the monetary payoff. For example, the most important to the players may be not to lose, while the amount of money won or lost can be of lesser importance. To take into account such a behavior the outcome of the game can be transformed into utility units by means of a utility function. In our example this transformation can have the following form: $-2 \rightarrow 0.0$, -1 $\rightarrow 0.0$, $0 \rightarrow 0.3$, $1 \rightarrow 0.9$, $2 \rightarrow 1.0$.

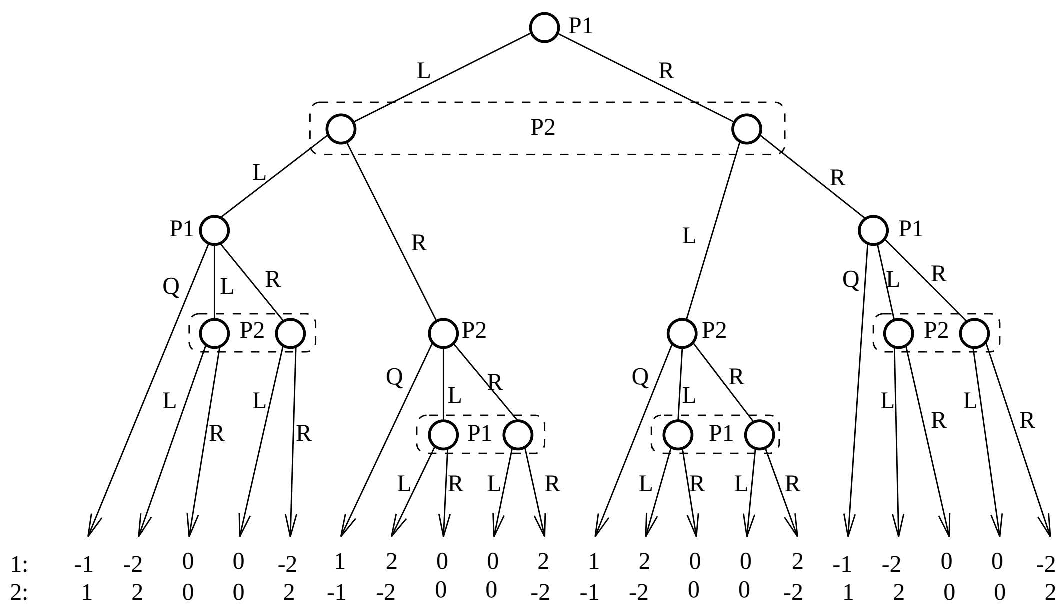

Figure C.1. *Game tree.*

Game environment

There are three basic pieces of information that a player can posses: the set of players, all actions available to the players, and all potential outcomes for all players.

In general a player can have complete or incomplete information about the game. Another aspect of the information availability to a player is the issue of forgetting. Perfect recall corresponds to a situation where the information set is consistent with a never-forgetting player. If this is not true, the game is classified as the one of imperfect recall.

The term common knowledge refers to the fact that all players have the same knowledge about the game. An example of common knowledge is a binding agreement which can be interpreted as a signed contract enforced by an outside authority. In general a binding agreement imposes restriction on the available actions on all players. A commitment is a particular case of binding agreement where one player restricts his actions and makes it common knowledge. A threat is a classical case of commitment.

C.1.3 Strategic form

The extensive form of presenting games is convenient for illustration of basic concepts from the game theory and some simple games but can be very unwieldy in many games due to a possible enormous number of nodes. In many cases this number can approach infinity. For this reason it may be convenient to describe the game in terms of each player's strategy. In this case a player's decision is a function of the information about the game available to the player. While the information set can be still very large (especially in the case of complete knowledge and perfect recall), the strategy description can be significantly simplified if only a part of information is important from the decision viewpoint.

While in general the strategic form is simpler and more natural than the extensive form, it is clear that this description suppresses the underlying game move structure. Two basic classes of strategies can be recognized: deterministic and probabilistic. In the first case the decision is a deterministic function of the available information. In the second case the choice has a random element which makes that based on the same information different decisions can be reached in different instances.

Equilibrium point

An equilibrium point is a combination of players' strategies for which each player maximizes his own utility by optimizing his own strategy, given the possible strategies of the other players. The equilibrium point was introduced by (Nash, 1950). Under certain assumptions it can be proven that all n-player non-cooperative games have at least one equilibrium point (Friedman, 1990). The solution uniqueness can be proven for some game classes (Friedman, 1990).

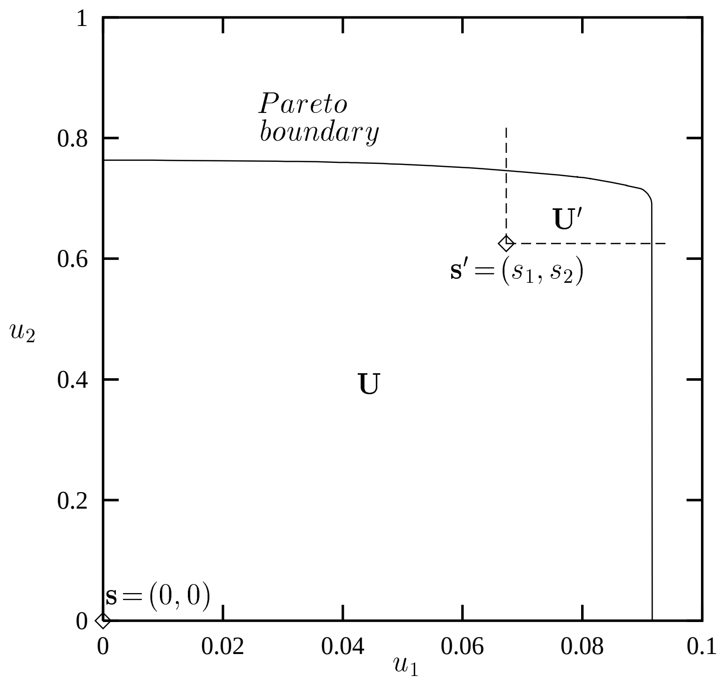

Figure C.2. *Example of bargaining domain.*

C.1.4 Coalitional form

The coalitional form assumes that any group of players can make contractual agreements. A group making a binding agreement is called coalition, C. Note that this form of game assumes certain cooperation among players. In such cooperative games the main focus of analysis are payoffs which can be achieved by each player or coalition. The strategies themselves are of lesser interest.

Characteristic function

The characteristic function describes the payoff possibilities for each coalition. This payoff, $v(C)$, represent the total utility that coalition C can obtain when their members cooperate.

Core

The core is a set of game solutions which are agreeable to all players. Here the term agreeable is strictly defined: agreeable game solution must give as much to any single player and to any potential coalition as they would get acting for themselves. The reason for this definition is that in general the game solution must be accepted by all players and potential coalitions. If any of the players or potential coalitions can achieve a better game outcome acting on their own, they will not agree to the proposed solution. In general the core can consist of many solutions, one solution, or can be empty. Nevertheless it should be indicated that the core consisting of one solution is unusual.

Bargaining domain and Pareto optimality

Since in cooperative games the final outcome is of main interest it is often convenient to analyze the domain of all possible outcomes, U, which is called bargaining domain. An outcome of the game is defined by the values of players utilities $\mathbf{u} = \{u_1, ..., u_n\} : \quad u_i \in R$. In general each utility can be expressed in different units. To simplify presentation in the following we consider the bargaining domains which are convex, closed, and bounded sets of $R^n : \quad R \geq 0$. An example of such a domain for $n = 2$ is given in Figure C.2.

A Pareto optimal solution is defined as a solution which ensures that there is no other solution in which a player can increase his utility without adversely affecting the other's. In Figure C.2 all solutions on the upper right boundary are Pareto optimal and form the Pareto boundary. The outcome of the game depends also on the starting point, $\mathbf{s} = \{s_1, ..., s_n\} : \quad \mathbf{s} \in U$, which is sometimes called the conflict or disagreement point. The starting point corresponds to a pre-game assumption that the game outcome, $\mathbf{u}^*$, cannot be worse for any player than the starting point, $\mathbf{u}^* \geq \mathbf{s}$. The starting point can limit the bargaining domain as shown in Figure C.2.

C.2 Cooperative vs. Non-cooperative Games

The principal difference between the cooperative and non-cooperative games is the introduction of binding agreements in cooperative games. For this reason the modeling and analysis of non-cooperative games focuses on optimization of players' strategies in order to achieve an equilibrium point. Due to the competitive nature of the bargaining process in non-cooperative games the outcome can be both unfair and non-efficient (below the Pareto boundary).

In the case of cooperative games, the possible agreement on the solution shifts focus on the features and properties of the solution which are often expressed in terms of axioms. The axioms try to incorporate the fairness features in the solution while efficiency is provided by required Pareto optimality. It should be underlined that there is no universal and objective fairness criteria and that there are several competing concepts, each of them having strong supporters. This indicates that the solution fairness judgment is a subjective process which may depend on a particular game formulation.

The analysis of cooperative games can also include the bargaining process. In this case the possibilities of binding agreement should be built into the game formulation. This approach was used in the original derivation of the Nash arbitrated solution for two-player cooperative games (Nash, 1950). Nevertheless, in the following we concentrate on the outcome of the cooperative games.

C.3 Different Formulations of Cooperative Games

Historically there are two formulations of cooperative games: One for two-player games and the other for n-player games with $n > 2$. The difference between the two is much deeper than the difference between the number of players. Let us consider first the n-player formulation ($n > 2$). The central issue in this formulation is the possibility of forming different coalitions with different sub-sets of players. While the final solution is achieved with cooperation of all players, the potential solutions for different possible coalitions influence the final outcome. This is evident in the already defined set of agreeable solutions (core). Concerning the choice of a particular solution from the core the potential coalitions are also important. This feature can be well illustrated by means of the Shapley value which can serve to find a unique solution.

To simplify presentation let us consider games with transferable utility, which means that the amount of utility achieved by a coalition can be divided among its members in any mutually agreeable fashion. In general the Shapley value, $\Phi(v) = \{\Phi_1(v), ..., \Phi_n(v)\}$, defines the payoff to each player. This payoff is a weighted average of the contributions that the player makes to the payoff of each coalition to which he belongs. The weight depends on the number of players, k, in each coalition. The value is given by

$$\Phi_i(v) = \sum_{C \in \mathbf{C}} [v(C) - v(C \setminus \{i\})] \frac{(k-1)!(n-k)!}{n!} \tag{C.1}$$

where $\mathbf{C}$ is the set of all coalitions. This solution can be also characterized by four natural conditions; for details see e.g. (Friedman, 1990).

Now it can be easily understood why there is significant difference between two and more player games. Simply there is only one coalition in a two-player game, which consists of both players, and there are no potential coalitions. As a result the proposed solutions for two-player games are analogous to a conflict which is resolved by an external arbiter according to certain rules which are believed to provide fairness and efficiency. Hence, the solution algorithms are called arbitrated schemes based on axioms. As previously indicated there is no objective measure of the fairness; thus different arbitration schemes can have appeal to different users and in different applications. This feature makes the arbitration schemes similar in concept to general laws governing society which are also constructed to be fair but are perceived differently by different people.

Although historically the arbitration schemes were developed for two-player games, the notion of external arbitration without considering the potential coalitions can be extended to a general n-player games. In fact some of the arbitration schemes developed for two-player games can be easily extended to the general case. This framework fits very well some of the problems considered in this book where the network control manager can be treated as an arbiter whose objective is to provide fair and efficient access to network resources for all users. In the remainder of this appendix we concentrate on a class of arbitration schemes which can be easily extended to the general case.

C.4 A Class of Arbitrated Solutions

In this section we discuss a class of arbitrated solutions derived by (Cao, 1982) for two-player games. This class is based on maximization of the product of player preference functions. This form of the objective function is associated with the Nash arbitrated solution. Originally Nash arbitration was defined by four axioms:

- *Symmetry*: If the bargaining domain is symmetric with respect to the axis $u_1 = u_2$ and the starting point is on this axis, then the solution is also on this axis.
- *Pareto optimality*: The solution is on the Pareto boundary.
- *Invariance with respect to utility transformations*: The solution for any positive affine transformation of $(U, \mathbf{s})$ denoted by $V(U)$, $V(\mathbf{s})$ is $V(\mathbf{u}^*)$ where $\mathbf{u}^*$ is the solution for the original system.
- *Independence of irrelevant alternatives*: If the solution for $(U_1, \mathbf{s})$ is $\mathbf{u}^*$, and $U_2 \subseteq U_1$, $\mathbf{u}^* \in U_2$, then $\mathbf{u}^*$ is also the solution for $(U_2, \mathbf{s})$. In other words this axiom states that if U_1 is reduced, in such a way that the original solution and starting point are still included, the solution of the new problem remains the same.

It is important that the Nash solution can be also expressed as maximization of the product of normalized utilities

$$\max_{u \in U}\{u'_1 \cdot u'_2\} \tag{C.2}$$

where u'_i is utility normalized by its maximum value in the utility domain U. Here the normalized utilities can be treated as the players' preference function which means that in the Nash solution each player is concerned only with his own gain. Cao extended this formulation to a family of preference functions which also takes into account the other player's gain. This family is defined as

$$v_1 = u'_1 + \beta(1 - u'_2) \tag{C.3}$$
$$v_2 = u'_2 + \beta(1 - u'_1) \tag{C.4}$$

where $\beta = [-1, 1]$ is a weighting factor. Using this definition of player's preference function one can define a class of arbitrated solutions as

$$u^* = \arg(\max_{u \in U}\{v_1 \cdot v_2\}) \tag{C.5}$$

Observe that for $\beta = 0$ the solution u^* corresponds to the Nash arbitration. For $\beta \neq 0$ the player's preference function takes into account the other player's utility. In particular for $\beta = 1$ the preference function treats with the same weight the player's gain and the other player's losses so the solution equalizes the normalized utilities of both players. For $\beta = -1$ both gains have the same weight so the solution maximizes the sum of normalized utilities. Interestingly these two solutions correspond to well known arbitration schemes. The first solution ($\beta = 1$) is equivalent to the Raiffa-Kalai-Smorodinsky (henceforth called Raiffa) solution (Raiffa, 1953; Kalai and Smorodinsky, 1975) which is based on four axioms. The first three axioms are the same as the ones for the Nash arbitration. The fourth is called *Monotonicity* and is defined as follows:

- *Monotonicity*: If $U_2 \subseteq U_1$ and $\max\{u_1 : \mathbf{u} \in U_1\} = \max\{u_1 : \mathbf{u} \in U_2\}$ and $\max\{u_2 : u \in U_2\} \leq \max\{u_2 : u \in U_1\}$, then $u_2^{2*} \leq u_2^{1*}$, where $\mathbf{u}^{j*}$ denotes the solution for $(U_j, \mathbf{s})$. In other words, for any subset of U_1 the solution for the second player cannot be improved if the maximum utility of the first player is constant.

The second solution ($\beta = -1$) is equivalent to the modified Thomson solution (Cao, 1982) which is defined by the utilitarian rule maximizing the sum of the normalized utilities.

In (Cao, 1982) it is shown that by changing β continuously from -1 to 1 one can achieve monotonically and continuously a set of solutions which relates $\beta \in [-1, 1]$ to a part of the Pareto boundary. This fact has an appealing geometrical interpretation. Namely for each β the solution is given by the tangent point between the Pareto boundary and a hyperbola from the set of hyperbolae defined by the function $v_1 \cdot v_2 = \text{const}$. As we are moving from $\beta = 0$ to $\beta = -1$ the branches of the hyperbola are widening to become a straight line normal to the line $u_1' = u_2'$ for $\beta = -1$. As we are moving from $\beta = 0$ to $\beta = 1$ the branches of the hyperbola are narrowing to become a semi-line $u_1' = u_2'$ for $\beta = 1$. This feature shows that although the Nash, Raiffa, and the modified Thomson arbitration schemes have very different characteristics they are special cases of a wider class of arbitration schemes. In this context it is also important to emphasize that in the preference space, $\{v_1, v_2\}$, all mentioned solutions become Nash solutions.

C.4.1 Extension to the general case

The simple mathematical form of the presented class of arbitrated solutions and its natural geometrical interpretation makes the extension to general n-player games relatively simple. Namely, instead of starting from axiomatic representation extensions for Nash, Raiffa and Modified Thomson solutions, one can try to generalize the mathematical representation of these solutions ensuring that the central features are preserved and that the extension also covers the case of two-player games. Generalization of Nash arbitration scheme is straightforward. In this case the solution is defined by

$$u^* = \arg\left(\max_{u \in U}\left\{\prod_{i=1}^{i=n} v_i\right\}\right) \tag{C.6}$$

where the preference function is the same as in the two-player game ($v_j = u_j'$). In (Stefanescu and Stefanescu, 1984; Mazumdar, Mason, and Douligeris, 1991) it was proved that this formulation of the Nash generalized arbitration scheme is equivalent to a formulation based on generalized axioms. This result is important for further analysis since the considered class of arbitration schemes is based on Nash solutions in preference function domains.

Assuming that the objective of the modified Thomson arbitration scheme is to maximize the sum of the normalized utilities, the generalization of this scheme is also straightforward and is defined by the following preference function:

$$v_j = \sum_i u'_i \tag{C.7}$$

The objective of the Raiffa arbitration scheme is to equalize the normalized utilities. This objective can be achieved by applying the preference function defined as

$$v_j = \sum_j u'_j + 1 - \frac{1}{n-1} \sum_{i \neq j} u'_i \tag{C.8}$$

This form indicates that in the generalized case of the Raiffa arbitration scheme the player's preference function treats with the same weight the player's gain and the average losses of the other players.

All three cases can be viewed as special instances of a set of preference functions defined by

$$v_j = u'_j + |\beta(n-1)| - \beta \sum_{i \neq j} u'_i \tag{C.9}$$

for $\beta = 0, -1, 1/(n-1)$ respectively. As in the case of two players, by changing β continuously from -1 to $1/(n-1)$ one can achieve continuously a set of solutions which relates $\beta \in [-1, 1/(n-1)]$ to a part of the Pareto boundary.

C.5 Discussion and Bibliographic Notes

Although in this appendix we introduced several basic concepts applicable to non-cooperative and cooperative game theory, the latter is the main focus of the presentation. Within the cooperative game theory one can recognize two areas of interest. The coalitional formulation of the game is important in the cases where the players have freedom to establish an agreement with any subgroup of players. Although the final solution requires consent from all players, the potential coalition agreements influence the final solution. The other formulation assumes that there is an external arbitrator who decides what the game outcome should be based on a set of axiomatic rules which can be compared to laws in a society. These rules should provide a fair and efficient game outcome and are accepted by all players. Historically this approach was developed mainly for two-player games since coalitional approach cannot be applied in this case. Nevertheless, the general n-player case is also of interest, especially in the cases where there are several players who want to maximize their outcome but because of the game nature they cannot form coalitions. In this case they have to resort to an external arbiter in order to achieve a fair and efficient payoff. This formulation fits very well into the problem of fair and efficient resource allocation to connection classes in telecommunication networks where the connection classes can be interpreted as players and the network control takes the role of an arbiter. For this reason we described a generalization of a class of arbitration schemes derived for two-player games. Application of this approach to network resource management is described in Chapter 8.

The reader interested in game theory can refer to a large literature on the subject. For example, a nice overview of the history and development of game theory is given in (Aumann, 1987). A survey of all important areas of game theory can be found in (Aumann, 1991). In (Aumann, 1985) the focus is on methodology, on the application of game theory to the real world, and on the objectives of the game theory. The presentation in this appendix follows (Friedman, 1990) except the section on the class of arbitrated solutions which is based on (Cao, 1982).

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